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Langevin dynamics, scale invariance, and granular flows

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We propose a dynamic model for granular flows. Starting from the equation of motion for a grain together with the equation of continuity, we derive a Langevin equation describing the time evolution of the granular system. We show how the model displays scale invariance and reduces to several existing models in appropriate limits. In particular, we also point out the possibility of different behaviors in granular systems according to their sizes.

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I. INTRODUCTION

In studies of granular materials, one of the most exciting challenges is to develop a sound model that can describe the complex dynamical behavior of granular particles. Granular particles flow like a liquid and fill any shape. Yet an attempt to model its dynamics by equations of motion describing a pure fluid is likely to fail because granular particles also exhibit solidlike behavior, enduring shear stress and supporting themselves with a finite angle of repose. These peculiar features have given rise to a number of recent investigations, both experimentally [1–3] and theoretically [4–9]. In particular, the idea of "self-organized criticality" introduced by Bak, Tang, and Wiesenfeld [4] has motivated much interest in the possibility of scale invariance in dynamical behavior of granular piles [1–9] and earthquakes [10]. Subsequently, by considering various symmetries of the system, Hwa and Kardar constructed a Langevin equation which appears to describe surface fluctuations of a sandpile in the hydrodynamic limit [5]. In this case, the scale invariance of the system has been shown to be a consequence of the conservation law and anisotropy [8].

In contrast with this theoretical prediction, experimental results of real sandpiles still remain unclear regarding scale invariance, even suggesting that the appearance of scale invariance is a finite-size effect [3]. Large sandpiles simply do not exhibit scale invariance but rather appear to oscillate between two fixed points [3,11]. We recognize that most of the existing models [5–9] have been constructed largely via phenomenological arguments and thus lack microscopic features which appear to be crucial in displaying dynamical behavior peculiar to granular systems.

The purpose of this paper is to propose a dynamic model for granular flows. Starting from the equation of motion for a grain together with the continuity equation, we derive a Langevin equation that describes the bulk fluctuations of the system. In the absence of gravity, granular flow is described by a relaxational dynamics: When driven by the conservative noise, the system relaxes to the equilibrium state characterized by the Ising Hamiltonian [6]. When driven by the nonconservative noise, the system relaxes to the critical state characterized by the Gaussian Hamiltonian at its criticality. In the presence of gravity, the proposed model predicts that the *nonlocal* nature of the interactions resulting from gravity can *destroy* the scale invariance even though the system possesses anisotropy and obeys a conservation law. This leads to the interesting possibility of different behaviors in granular systems according to their sizes: for large sandpiles, nonlocal nature is important and the mass term appears in the Langevin equation, thus destroying scale invariance. In the absence of nonlinear and nonlocal terms, this model is shown to reduce to the diffusing void model [9]. It also reduces to the Hwa-Kardar equation [5] in the appropriate limit, thus making the nontrivial prediction that the surface fluctuations and bulk fluctuations are of the same nature, a result suggested by numerical simulations of sandpile models [5].

II. LANGEVIN DYNAMICS

The conventional approach has been to start from the continuity equation together with momentum conservation. However, there is little we can do in finding out the relation between the stress and the strain tensor for granular systems. The plasticity model has long been used to remedy this deficiency. However, it does not describe even the most simple flow patterns of granular flows. In this paper, we drastically depart from this conventional approach and present a different one.

Consider the number density $\rho(\mathbf{r},t)=\sum_{\alpha=1}^{N}\delta(\mathbf{r}-\mathbf{r}_\alpha(t))$, where $\mathbf{r}_\alpha$'s are the position of the grains and N is the total number of grains in the system. In the stationary state, $d\mathbf{r}_\alpha/dt=\mathbf{v}(\mathbf{r}_\alpha)$, and one can easily derive the continuity equation for $\rho(\mathbf{r},t)$; $\partial_t\rho(\mathbf{r},t)+\nabla\cdot(\rho\mathbf{v})=0$. Consider now the coarse-grained density $n(\mathbf{r},t)$ defined as the total number of grains in a box of size b^d. Then it is straightforward to show that $n(\mathbf{r},t)$ also satisfies the continuity equation:

$$\frac{\partial}{\partial t}n(\mathbf{r},t)+\nabla\cdot\mathbf{j}(\mathbf{r},t)=0 , \tag{1}$$

with $\mathbf{j}(\mathbf{r},t)\equiv n\mathbf{v}$ the current density. Now the motion of a grain will be governed by the total force on it, which in general consists of three parts: the applied force $\mathbf{F}$ including the normal contact force and gravity, the frictional force $\mathbf{f}$, and the "random force" $\boldsymbol{\eta}$ which may result from, e.g., random shaking.

In the stationary state, the velocity of a grain does not change in time, and we impose the balance condition: $\mathbf{F}+\mathbf{f}+\boldsymbol{\eta}=0$. In principle, this condition gives the velocity $\mathbf{v}$ or the current density $\mathbf{j}=n\mathbf{v}$ as the function of the number density $n(\mathbf{r},t)$, which, upon substituting in the continuity equation, leads to the desired Langevin equation for the system. We now derive the general form for $\mathbf{j}$ from the following microscopic consideration. First, the normal force on the grain at $\mathbf{r}$ is exerted by its neighboring grains located in the direction opposite to the force, and has the components $F_x=-\alpha[n(\mathbf{r}+\hat{\mathbf{x}})-n(\mathbf{r}-\hat{\mathbf{x}})]$, where α is a constant measuring the strength of the force and the space has been rescaled in units of the intergranular spacing. Thus in the long-wavelength limit, we obtain the normal force $\mathbf{F}=-\alpha\nabla n$. Next, the frictional force, which is exerted by neighboring grains located in the direction perpendicular to the force, has the direction opposite to the velocity $\mathbf{v}$. For small $\mathbf{v}$, its magnitude is expected to be proportional to that of $\mathbf{v}$. Therefore, to the lowest order, we write $f_x=-\gamma v_x[n(\mathbf{r}+\hat{\mathbf{y}})+n(\mathbf{r}-\hat{\mathbf{y}})+n(\mathbf{r}+\hat{\mathbf{z}})+n(\mathbf{r}-\hat{\mathbf{z}})]$ or $\mathbf{f}=-4\gamma n\mathbf{v}$ in the long-wavelength limit. The balance condition then leads to the current density $\mathbf{j}=-\alpha_1\nabla n+\epsilon\boldsymbol{\eta}$ with $\alpha_1\equiv\alpha/4\gamma$ and $\epsilon\equiv 1/4\gamma$. Substituting it into the continuity equation, we derive the diffusion equation describing the random walk [9]. Note that the friction law used above does not take into account the threshold term necessary to move the locked particle. In the absence of gravity, the force exerted on the grain is only through normal contact and if the net force vanishes, then the grain should not move since the friction alone cannot cause the grain to do so [16]. Such a model was recently proposed [9], where particles move under the influence of gravity through a vacancy motion and the locking as well as the momentum transfer among particle has been ignored. In this case, friction is due to the contact with grains and thus it might just be proportional to the velocity. In this respect, the random walk model of granular flows [9] essentially corresponds to the linear limit of the model proposed here. In spite of its simplicity, this model has reproduced many of the unique features of granular flows including the evolution of free surface and the stream lines around an obstacle. Obviously, other higher-order contributions to $\mathbf{F}$ and $\mathbf{f}$ will come from next-nearest neighbors. Without going into detail, one may construct the general form of $\mathbf{j}$ through the use of the symmetry of the system [5]. At this point, it might be convenient to define a new scalar field $\psi(\mathbf{r},t)\equiv 2n(\mathbf{r},t)-\Lambda$, where Λ is a constant that measures the number of (fine-grained) particles in a (coarse-grained) grain [12]. We then extend the range of ψ to the entire real axis, and write down the Langevin equation in terms of ψ:

$$\frac{\partial\psi}{\partial t}=\tilde{\alpha}_1\nabla^2\psi+\tilde{\alpha}_2\nabla^2(\psi^2)+\cdots-\tilde{\beta}_1\nabla^2(\nabla^2\psi)-\cdots+\zeta , \tag{2}$$

with suitably renormalized coupling constants. The noise term ζ is related to the random force $\boldsymbol{\eta}$ via $\zeta(\mathbf{r},t)\equiv-(\epsilon/2)\nabla\cdot\boldsymbol{\eta}(\mathbf{r},t)$. In the presence of noise, the diffusion equation has been studied with regard to the possibility of scale invariance [8]. Its dynamics in general depends in a crucial way on the nature of the noise ζ, which is in turn related to the random force $\boldsymbol{\eta}$: the first one is the random force with no spatial correlations $\langle\eta_i(\mathbf{r},t)\eta_j(\mathbf{r}',t')\rangle=D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\delta_{ij}$, for which the system described by (2) relaxes to an equilibrium state characterized by the Ising Hamiltonian,

$$\mathcal{H}=\int d^3r\left[\tfrac{1}{2}\tilde{\alpha}_1\psi^2+\tfrac{1}{3}\tilde{\alpha}_2\psi^3+\tfrac{1}{4}\tilde{\alpha}_3\psi^4+\tfrac{1}{2}\tilde{\beta}_1(\nabla\psi)^2-h\psi\right] ,$$

where an external magnetic field h controls the average value of ψ or the number of grains. The cubic term can be eliminated by a shift of ψ, resulting in a modification of h; higher-order terms are irrelevant [13]. Thus the system driven by the conservative noise eventually relaxes to the stationary state, where correlations in general decay exponentially in space although they decay algebraically in time, implying the absence of scale invariance. In a different context, such an Ising model has been also proposed for the equilibrium description of a granular system [6,14]. The second one is that characterized by algebraic correlations $\langle\eta_i(\mathbf{r},t)\eta_j(\mathbf{r}',t')\rangle=(D/4\pi)|\mathbf{r}-\mathbf{r}'|^{-1}\delta(t-t')\delta_{ij}$ which leads to the *nonconservative* noise with no spatial correlations $\langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle=\tilde{D}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$ and the corresponding Langevin equation again gives the stationary state described by the Gaussian Hamiltonian [13] at its criticality, $\mathcal{H}=\int d^3r\tfrac{1}{2}\tilde{\alpha}_1(\nabla\psi)^2$. Consequently, correlations decay algebraically both in space and in time with Gaussian exponents, leading to generic scale invariance [15].

III. EFFECTS OF GRAVITY

Gravity introduces anisotropy in the system and affects the dynamics in a significant way. To be specific, let us first consider the normal force in the presence of gravity $\mathbf{g}=-g\hat{\mathbf{z}}$. The downward normal force acting on a particular grain comes from the weight of those grains above it. This downward normal force together with its own gravitational force will be balanced with the upward normal force exerted by the neighboring grain just below if it is present. We now derive the z component of the force acting on the grain at $\mathbf{r}$. As shown in Fig. 1, the force acting on the grain at the center (black) consists of two parts: normal force $\mathbf{F}_b$ acting from below and the force from above due to the piles of grains, $\mathbf{F}_a$. Now the total force F_z acting on the grain at the center is *zero* if there is

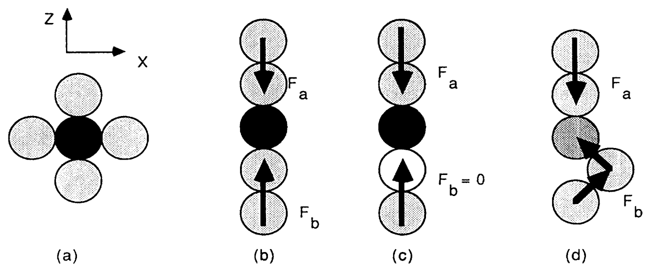

FIG. 1. (a) Forces acting on the grain at the center (black) located at position $\mathbf{r}$. It is surrounded by neighbor grains located at $\mathbf{r}+\hat{\mathbf{z}}$ and $\mathbf{r}-\hat{\mathbf{z}}$ along the z axis and $\mathbf{r}+\hat{\mathbf{x}}$ and $\mathbf{r}-\hat{\mathbf{x}}$ along the x axis and $\mathbf{r}+\hat{\mathbf{y}}$ and $\mathbf{r}-\hat{\mathbf{y}}$ along the y axis, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along x, y, and z axes, respectively. Total force acting along the z axis has two components; $\mathbf{F}_a$ is the force due to the piles above it, and $\mathbf{F}_b$ is the normal force acting from below. (c) If a grain at $\mathbf{r}-\hat{\mathbf{z}}$ is missing, then $\mathbf{F}_b=0$ in the order n. (d) However, it might not be zero in the order n^2. As shown in the figure, the normal force can be through next-nearest neighbors, which is order n^2.

a grain that supports upward from below. In this case, $\mathbf{F}_a$ and $\mathbf{F}_b$ exactly balance out. If not, then the total force acting on the center grain will be just the total mass *above it* multiplied by the gravitational constant g. Thus we may write

$$F_z=-mg[1-n(\mathbf{r}-\hat{\mathbf{z}})]\times[1+n(\mathbf{r}+\hat{\mathbf{z}})+n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}})+n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}})n(\mathbf{r}+3\hat{\mathbf{z}})+\cdots] , \quad (3)$$

where $\hat{\mathbf{z}}$ is a unit vector along the z axis. Let us now coarse grain the system, and consider the hydrodynamic limit. Then we may expand the density such that

$$n(\mathbf{r}-\hat{\mathbf{z}})\approx n(\mathbf{r})-\frac{\partial n}{\partial z}+\cdots \quad (4)$$

and set

$$\Sigma_z(n)\equiv n(\mathbf{r}+\hat{\mathbf{z}})+n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}})+n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}})n(\mathbf{r}+3\hat{\mathbf{z}})+\cdots . \quad (5)$$

The z component of the total force acting on the grain, F_z, becomes

$$F_z=-\alpha_z[1-n(\mathbf{r}-\hat{\mathbf{z}})]\left[1+\sum_{p=1}^{L-z}n(\mathbf{r}+p\hat{\mathbf{z}})\right]\approx-\alpha_z\left[1-n+\frac{\partial n}{\partial z}\right][1+\Sigma_z(n)] , \quad (6)$$

where $\alpha_z\equiv mg$ is the weight of a particle, L is the (linear) size of the system. We also used the fact that upon coarse graining, $\Sigma_z n\approx\sum_{p=1}^{L-z}n(\mathbf{r}+p\hat{\mathbf{z}})$. We next consider the horizontal component F_x, which has again two components, the contact force from the left and right grains. But in this case, considering the random nature of the grain packing, we note that one must multiply the total mass *acting on* the right or left grains. So, we find for the x component of the normal force,

$$F_x=-\alpha_x\left[n(\mathbf{r}+\hat{\mathbf{x}})\sum_{p=1}^{L-z}n(\mathbf{r}+\hat{\mathbf{x}}+p\hat{\mathbf{z}})-n(\mathbf{r}-\hat{\mathbf{x}})\sum_{p=1}^{L-z}n(\mathbf{r}-\hat{\mathbf{x}}+p\hat{\mathbf{z}})\right] \quad (7a)$$

$$\approx-\alpha_x[n(\mathbf{r}+\hat{\mathbf{x}})-n(\mathbf{r}-\hat{\mathbf{x}})]\Sigma(n(\mathbf{r}+k\hat{\mathbf{z}}))-\alpha_x[n(\mathbf{r}+\hat{\mathbf{x}})+n(\mathbf{r}+\hat{\mathbf{x}})]\Sigma\left[\frac{\partial n}{\partial x}\right] \quad (7b)$$

$$\approx-2\alpha_x\frac{\partial n}{\partial x}\Sigma_z(n)-\alpha_x\left[\frac{\partial^2 n}{\partial x^2}+2n\right]\Sigma_z\left[\frac{\partial n}{\partial x}\right] \quad (7c)$$

$$\approx-2\alpha_x\frac{\partial n}{\partial x}\Sigma_z(n) , \quad (7d)$$

where we have ignored the higher-order terms. Note that the first term and the second term in Eq. 7(a) are the forces acting from the right and left, respectively. The physical meaning of it is as follows: first, the density terms, $n(\mathbf{r}+\hat{\mathbf{x}})$ and $n(\mathbf{r}-\hat{\mathbf{x}})$, are due to the fact that the contact is possible only when the nearest-neighbor site is occupied; second, the total contact force acting from the left (or right) must be proportional to the total mass *acting on* the nearest-neighbor grains.

Similarly, we find, for the y component,

$$F_y\approx-\alpha_y\frac{\partial n}{\partial y}\Sigma_z(n) . \quad (8)$$

If the packings of the grains are random, then α_x and α_y are of the same order. Thus we set $\alpha_x\approx\alpha_y\equiv\frac{1}{2}\alpha_\perp$ with $\alpha_\perp\approx\alpha_z$. These expressions can be combined to give the final expression for the force acting on a grain, which *increases* with the depth of the grain:

$$\mathbf{F}=-[\alpha_\perp\nabla_\perp n+\alpha_z\partial_z n\hat{\mathbf{z}}+\alpha_z(1-n)\hat{\mathbf{z}}]\Sigma_z(n) , \quad (9)$$

with $\partial_z \equiv \partial/\partial z$ and $\nabla_\perp \equiv \hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y$. If we consider the contribution coming from the next-nearest neighbors, then the second term in Eq. (3) is modified:

$$F_z = -mg[1-n(\mathbf{r}-\hat{\mathbf{z}})-n(\mathbf{r}+\mathbf{x}-\hat{\mathbf{z}})n(\mathbf{r}-2\hat{\mathbf{z}}) - n(\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{z}})n(\mathbf{r}-2\hat{\mathbf{z}})-\cdots] \times[n(\mathbf{r}+\hat{\mathbf{z}})+n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}}) + n(\mathbf{r}+\hat{\mathbf{z}})n(\mathbf{r}+2\hat{\mathbf{z}})n(\mathbf{r}+3\hat{\mathbf{z}})+\cdots] ,$$

which introduces a term, second order in n, so that Eq. (9) becomes

$$\mathbf{F} = -[\alpha_\perp \nabla_\perp n + \alpha_z \partial_z n \hat{\mathbf{z}} + \alpha_z(1-n-n^2)\hat{\mathbf{z}}]\Sigma_z(n) . \quad (10)$$

Our next task is to find the frictional force $\mathbf{f}$. The most commonly used form for the frictional force [16] is

$$\mathbf{f} = -(a+bv^2)\hat{\mathbf{v}} . \quad (11)$$

A. Small system

For a small system, the frictional force is proportional to the normal force and thus we expect

$$\mathbf{f} = -\Sigma_z(n)(a+bv^2)\hat{\mathbf{v}} . \quad (12)$$

From the balance equation $\mathbf{F}_{\text{ext}}+\mathbf{f}+\boldsymbol{\eta}=0$ we find

$$(a+bv^2)\hat{\mathbf{v}} = (\Sigma_z(n))^{-1}[\mathbf{F}_{\text{ext}}+\boldsymbol{\eta}] \equiv \mathbf{F} \equiv F\hat{\mathbf{v}} ,$$

from which we obtain the expression for the flux $\mathbf{j}$,

$$\mathbf{j} = n\mathbf{v} = nv\hat{\mathbf{v}} = n[(F-a)/b]^{1/2}\hat{\mathbf{v}} = \frac{n}{[b(F-a)]^{1/2}}\left[1-\frac{a}{F}\right]\mathbf{F} ,$$

with $F=|\mathbf{F}|$.

Now, let

$$a(n) = a_0 + a_1 n + a_2 n^2 + \cdots ,$$
$$b(n) = b_0 + b_1 n + b_2 n^2 + \cdots .$$

Recall now the expression for $\mathbf{F}$ [Eq. (10)] and the definition of the density $n \equiv (\Lambda+\psi)/2$. Then,

$$a(\psi) = a_0' + a_1'(\psi) + \cdots ,$$
$$b(\psi) = b_0' + b_1'(\psi) + \cdots ,$$

and

$$F \equiv |\mathbf{F}| = \alpha_z\left[1-\frac{\Lambda}{2}\right][1+(\partial_z\psi-\psi)+O(\psi^2)] .$$

Thus we find

$$\mathbf{j} = \frac{(\Lambda+\psi)/2}{\{(b_0'+b_1'\psi+\cdots)[\alpha_z(1-\Lambda/2)][1+(\partial_z\psi-\psi)]-(a_0'+a_1'\psi+\cdots)\}^{1/2}} \times\left[1-\frac{a_0'+a_1'\psi+\cdots}{\alpha_z(1-\Lambda/2)[1+(\partial_z\psi-\psi)+\cdots]}\right]\mathbf{F} \equiv C\mathbf{F} ,$$

with

$$C \equiv \frac{\Lambda}{2\{b_0'[\alpha_z(1-\Lambda/2)-a_0']\}} \times\left[1-\frac{a_0'}{\alpha_z(1-\Lambda/2)}\right] = \text{const} .$$

Hence we derive

$$\mathbf{j} \equiv n\mathbf{v} = -\alpha_\perp\nabla_\perp\psi - \alpha_z\partial_z\psi\hat{\mathbf{z}} + \lambda_1\psi\hat{\mathbf{z}} + \lambda_2\psi^2\hat{\mathbf{z}} , \quad (13)$$

where all the coupling constants have been renormalized and rescaled with appropriate scale factors. The constant term as well as higher-order terms except the square term are irrelevant and have been neglected. Interestingly, (13) predicts the well-known fact that the velocity $\mathbf{v}$ of a grain does not increase with the depth of the granular system. This clearly originates from the fact that the frictional force as well as the normal force increases with depth, and is in sharp contrast with a fluid system, where the flow velocity increases with depth. Putting Eq. (13) together with the random force $\boldsymbol{\eta}$ back into the continuity equation (1), we arrive at the desired Langevin equation:

$$\frac{\partial\psi}{\partial t} = -\nabla\cdot\mathbf{j} = \alpha_\perp\nabla_\perp^2\psi + \alpha_z\partial_z^2\psi - \lambda\partial_z(\psi^2) + \zeta , \quad (14)$$

where the linear term $\partial_z\psi$ has been omitted since it can be removed by the Galilean transformation $z \to z - vt$ with suitable v [17]. The noise term in the above equation might be set to zero.

Note that in the absence of the nonlinear term ψ^2 and noise term ζ and with $\alpha_z=0$, Eq. (14) is precisely the governing equation of the diffusing void model if we put back the $\partial_z\psi$ term. We thus *have derived* the diffusing void model [9] from microscopic consideration. The precise characteristic of the noise term ζ included in (14) cannot be derived from that of the random force $\boldsymbol{\eta}$ since the transfer rule of a random force under gravity is unclear. Nevertheless, for the generic random force, it is reasonable to assume that the noise correlations are given by

$$\langle \zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle = [D_1 + D_2\nabla^2 + \cdots]\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') . \quad (15)$$

Remarkably, Eq. (14) is precisely the Langevin equation assumed to describe the sandpile problem with $\psi(\mathbf{r},t)$ describing the height profile of the sandpile [5]. That equation has been studied in detail to reveal scale invariance. It is of interest to note that in the sandpile problem $\psi(\mathbf{r},t)$ is defined on the surface and consequently, the dimension of the problem is $d-1$ for the d-dimensional sandpile. Here, on the other hand, $\psi(\mathbf{r},t)$ represents the number density and is defined not only on the surface but also throughout the entire system, making the dimension d. This strongly suggests that the surface fluctuations and bulk fluctuations in a granular system are of the same nature, as also indicated by numerical simulations of Hwa and Kardar [5].

B. Large system

For a large system, the normal force is large and thus we expect that the friction presumably tends to saturate and eventually becomes independent of the normal force (Fig. 2). In this *fluid* regime, the friction has the form

$$\mathbf{f}=-(a+bv^2)\hat{\mathbf{v}} ,$$

while the normal force still increases with depth. From the balance equation, we find

$$(a+bv^2)\hat{\mathbf{v}}=\mathbf{F}_{\rm ext}+\boldsymbol{\eta}\equiv\mathbf{F}\equiv F\hat{\mathbf{v}} ,$$

where

$$\mathbf{F}=-[\alpha_\perp\nabla_\perp n+\alpha_z\partial_z n\hat{\mathbf{z}}+\alpha_z(1-n)\hat{\mathbf{z}}]\Sigma_z(n)$$

and

$$F=|\mathbf{F}|=\Sigma_z(n)\alpha_z(1-\Lambda/2)[1+(\partial_z\psi-\psi)+\cdots] .$$

Hence we find the different expression for $\mathbf{j}$,

$$\mathbf{j}=\frac{\Lambda}{\{2b_0'[\alpha_z(1-\Lambda/2)\Sigma_z(n)-a_0']\}^{1/2}}\times\left[1-\frac{a_0'}{\alpha_z(1-\Lambda/2)\Sigma_Z(n)}\right]\mathbf{F} .$$

Since $n=(\Lambda+\psi)/2$, the summation at the end of the above equation becomes

$$\Sigma_z(n)\approx\int_z^L dz\, n=\int_z^L dz\frac{\Lambda+\psi}{2}=\frac{\Lambda(L-z)}{2}+\tfrac{1}{2}\int_z^L dz\,\psi \equiv\lambda+\int_z^L dz\,\psi/2 . \qquad (16)$$

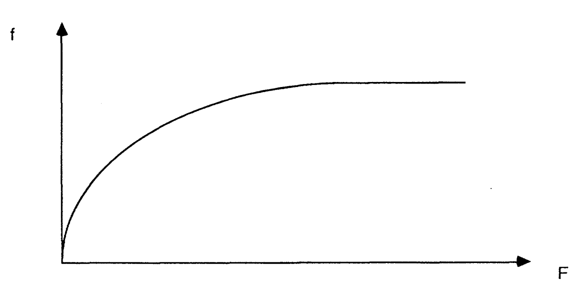

FIG. 2. Friction force $\mathbf{f}$ as a function of normal force $\mathbf{F}$.

Choose $2\lambda\gg\int dz\,\psi$ and then expand ψ in the hydrodynamic limit, which leads to the following current density:

$$\mathbf{j}=-\alpha_\perp\nabla_\perp\psi-\alpha_z\partial_z\psi\hat{\mathbf{z}}+\lambda_1\psi\hat{\mathbf{z}}+\cdots -(\mu_0+\mu_1\psi+\cdots)\hat{\mathbf{z}}\int_z^L\psi\,dz , \qquad (17)$$

where $\mu_0\ldots$ are appropriately scaled constants. Note that for a sufficiently large system, $\Sigma_z(n)\gg a_0'$ and in this limit,

$$\mathbf{j}=n\mathbf{v}\approx\frac{\Lambda}{[2b_0'\alpha_z(1-\Lambda/2)\Sigma_z(n)]^{1/2}}\mathbf{F} \approx-\frac{\Lambda}{[2b_0'\alpha_z(1-\Lambda/2)]^{1/2}}\times[\alpha_\perp\nabla_\perp n+\alpha_z\partial_z n\hat{\mathbf{z}}+\alpha_z(1-n)\hat{\mathbf{z}}][\Sigma_z(n)]^{1/2} .$$

In the fluid regime, $n\approx$const. Hence we have

$$\Sigma_z(n)=\int_z^L dz\,n\approx L-z\equiv h \quad \text{(depth)} .$$

Therefore the current density $|\mathbf{j}|\approx|\mathbf{v}|\approx\sqrt{h}$ and we recover the well-known result $v\approx\sqrt{h}$, i.e., the velocity is proportional to the square root of the height for the fluid regime. Putting Eq. (17) back into the continuity equation (1), we obtain the corresponding Langevin equation, which now contains a *mass term*, $-\mu_0\psi$:

$$\frac{\partial\psi}{\partial t}=-\mu_0\psi+\alpha_\perp\nabla_\perp^2\psi+\alpha_z\partial_z^2\psi-\partial_z\psi^2+\zeta . \qquad (18)$$

The mass term destroys scale invariance and leads to exponential decay with the characteristic time μ_0^{-1}. Therefore the system does not display scale invariance *in spite of anisotropy and the apparent conservation law* in the starting equation of continuity [Eq. (1)]. This seems to provide a natural explanation as to the key difference between a granular system and a fluid system in spite of their many similarities: The former can display scale invariance while there is no scale invariance in the latter. Of particular interest here is the possibility of different behaviors in granular systems of different sizes. In a sufficiently small granular system, the normal force acting on the grain even at the bottom will be small, leading to the friction proportional to the normal force. Then the system is described by Eq. (14), and exhibits scale invariance. In a large granular system, on the other hand, the friction acting on most of the grains will saturate due to the large normal force [11]. Consequently, the system is in the fluid regime and described by Eq. (18), leading to exponential relaxation. This may provide some clue to the striking difference according to their sizes which has been indeed observed in experiments of sandpiles [3].

In summary, we have derived a nonlinear evolution equation for the granular flows from microscopic considerations, which reduces to the diffusing void model [9] and the nonlinear Langevin equation derived by Hwa and Kardar [5] and the Ising model description of Mehta and Edwards [6]. The model we have derived, however, is not complete and may require further modifications along the following direction. First, the evolution equation has been derived in an infinite system with no walls present. It is certain that the walls take the load and might complicate the dynamics of the granular system. In fluids, the walls provide boundary conditions to the second-order Navier-Stokes equation. One may impose similar boundary conditions to the evolution equation derived in this paper, say, the reflecting boundary condition [9]. However, the detailed motion of the grains near the boundary might be complex and needs exploration from a microscopic point of view such as the molecular-dynamics simulations [18]. Note that our equation does not contain the temperature field for the obvious reason that thermal energy is too small to trigger the motion in the granular system [19]. Second, the most serious question concerns the stress distribution of the granular systems [20], for which case the coarse-grained description might not be so useful. Nevertheless, we anticipate that the nonlinear evolution equations derived in this paper might provide quite useful information regarding the complex dynamics displayed by granular systems.

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